On some quadratic algebras

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Abstract

We study some quadratic algebras which are appeared in the low–dimensional topology and Schubert calculus. We introduce the Jucys–Murphy elements in the braid algebra and in the pure braid group, as well as the Dunkl elements in the extended affine braid group. Relationships between the Dunkl elements, Dunkl operators and Jucys–Murphy elements are described.

1 Introduction

In this paper we study some quadratic algebras which are naturally appeared in the low–dimensional topology [B-N], [Dr2], [Ko2] and Schubert calculus [FK], and investigate some of their properties. We define a quadratic algebra $\mathcal{G}_n$, which is a further generalization of the quadratic algebras $\mathcal{E}_n$ and $\mathcal{E}_n^t$ introduced in [FK], Sections 2 and 15. Our main idea is to apply the results obtained for the braid algebra $\mathcal{B}_n$, [Dr2], [Ko2], to the algebra $\mathcal{G}_n$. The main observation is that some results which are well–known for the braid algebra $\mathcal{B}_n$, can be (re)proven for the quadratic algebra $\mathcal{G}_n$. For example, it happens that the quadratic algebra $\mathcal{G}_n$ and the braid algebra $\mathcal{B}_n$ have the same Hilbert series

$$H(\mathcal{B}_n;t)=H(\mathcal{G}_n;t)=\frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)},$$

and have the commutative subalgebras $\mathcal{K}_n\subset\mathcal{B}_n$ and $\mathcal{H}_n\subset\mathcal{G}_n$ which are both isomorphic to the ring of polynomials. The algebra $\mathcal{K}_n$ is generated by the Jucys–Murphy elements, whereas the algebra $\mathcal{H}_n$ is generated by the Dunkl ones. Both algebras $\mathcal{G}_n$ and $\mathcal{B}_n$ have the additive bases consisting of all monomials in normal

form (Theorem 2.4 and Corollary 4.3; cf. [L]). We expect also that the certain quotients $\mathcal{E}_n^0$ and $\mathcal{B}_n^0$ of the quadratic algebras $\mathcal{G}_n$ and $\mathcal{B}_n$ respectively, have the same Hilbert polynomials as well (see Section 9 for details).

One of the main goals of this paper is to describe the relationships between the Dunkl and Jucys–Murphy elements, and to compute the dimensions of homogeneous component of degree ≤ 6 of the quadratic algebra $\mathcal{E}_n^0$.

The structure of the paper is following:

In Section 2 we define the braid algebra $\mathcal{B}_n$ as the infinitesimal deformation of the pure braid group P_n (cf. T. Kohno [Ko2]). The algebra $\mathcal{B}_n$ and its completion $\widehat{\mathcal{B}}_n$ have many important applications to the low–dimensional topology, see e.g. [B-N], [Dr2], [F], [Ko3], [L].

In Section 3 we define the Jucys–Murphy elements d_k in the braid algebra $\mathcal{B}_n$, and the multiplicative Jucys–Murphy elements D_k in the pure braid group P_n. Follow to A. Ram, [Ra], we prove that the Jucys–Murphy element $d_k \in \mathcal{B}_n$ is the quasi–classical limit of the element $D_k \in P_n$. We prove also, that the infinitesimal deformation of the multiplicative Jucys–Murphy element D_k coincides with the element d_k.

In Section 4 we define the quadratic algebra $\mathcal{G}_n$ and compute the Hilbert series for this algebra. In Section 5 we define the Dunkl elements θ_j in the quadratic algebra $\mathcal{G}_n$ and describe a commutative subalgebra generated by θ_j, $1 \leq j \leq n$.

In Section 6, which contains one of the main results of this paper, we study a relationship between the Jucys–Murphy elements and the Dunkl elements. We define the dual Dunkl elements $Y_1^*, \ldots Y_n^*$ as certain elements in the extended affine braid group $\widetilde{B}_n$, and prove that under a natural homomorphism $\pi : \widetilde{B}_n \to B_n$, the dual Dunkl element Y_k^* maps to the multiplicative Jucys–Murphy element D_k. We explain also a connection between Dunkl element $Y_k \in \widetilde{B}_n$ and the Dunkl operator $\mathbf{Y}_k \in \mathcal{D}_{q,x}[W]$, where $\mathcal{D}_{q,x}[W]$ is the algebra of q–difference operators with permutations.

In Section 7 we study the relations in the algebra $\mathcal{E}_n^t$. In Section 8 we use the properties of "normal basis" introduced in Section 4, to study the quotient algebra $\mathcal{E}_n^0$, and compute the dimensions of homogeneous components $\mathcal{E}_{n,k}^0$, $1 \leq k \leq 6$, and also the Hilbert polynomial $H(\mathcal{E}_4^0;t)$. The last polynomial was also computed, using computer, by J.-E. Ross [Ro], who computed the Hilbert polynomial $H(\mathcal{E}_5^0;t)$ as well.

In Section 9 we consider a certain quotient $\mathcal{B}_n^0$ of the braid algebra $\mathcal{B}_n$, and make a conjecture that the quadratic algebra $\mathcal{E}_n^0$ and $\mathcal{B}_n^0$ have the same Hilbert polynomials. In particular, we expect that the algebra $\mathcal{B}_n^0$ (as well as $\mathcal{E}_n^0$) has a finite dimension.

In Section 10 we study the commutative quadratic algebra $\mathcal{A}_n^t$ (denote by $\mathcal{E}C_n$ in [FK], Section 4.3), which is a commutative quotient of $\mathcal{E}_n^t$ (see Definition 10.1), and explain some details which were omitted in [FK].

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2 Quadratic algebra $\mathcal{B}_n$

We start with consideration of the quadratic algebra $\mathcal{B}_n$ which is the infinitesimal deformation of the pure braid group P_n, [Ko2]. This quadratic algebra is well–known as the algebra of (finite order) Vassiliev invariants of braids, [Ko3], [F], [B-N]. The completion $\widehat{\mathcal{B}}_n$ of $\mathcal{B}_n$ with respect to the grading was considered in the papers of V. Drinfeld, [Dr1], [Dr2], in his study of quasi–Hopf algebras, and in the papers of T. Kohno, [Ko1]–[Ko3], in his study of monodromy representations of braid groups.

Definition 2.1 ([Ko2], [Dr2]) *Define the braid algebra $\mathcal{B}_n$ as the quadratic algebra (say over $\mathbb{Z}$) with generators X_{ij}, $1 \le i < j \le n$, which satisfy the following relations*

$$1)\quad X_{ij}\cdot X_{jk}-X_{jk}\cdot X_{ij}=X_{ik}\cdot X_{ij}-X_{ij}\cdot X_{ik}=X_{jk}\cdot X_{ik}-X_{ik}\cdot X_{jk}; \tag{2.1}$$

$$2)\quad X_{ij}\cdot X_{kl}=X_{kl}\cdot X_{ij},\quad \text{if all } i,j,k,l \text{ are distinct.} \tag{2.2}$$

Remark 2.2 The algebra $\mathcal{B}_n$ is denoted by $\mathcal{P}_n$ in [L], and as A^n in [Ko1]–[Ko3].

The algebra $\mathcal{B}_n$ has a natural structure of a cocommutative Hopf algebra. Namely, a comultiplication $\Delta:\ \mathcal{B}_n \to \mathcal{B}_n \otimes \mathcal{B}_n$, antipode $S:\ \mathcal{B}_n \to \mathcal{B}_n$, and counit $\epsilon:\ A \to \mathbb{Z}$, can be define as follows

$$\begin{aligned}\Delta(X_{ij}) &= 1\otimes X_{ij}+X_{ij}\otimes 1,\\ S(X_{ij}) &= -X_{ij},\\ \epsilon(X_{ij}) &= 0,\quad \epsilon(1)=1.\end{aligned}$$

Let us explain briefly an origin of the relations (2.1)–(2.2). For more details and proofs see [Dr1], [Dr2], [F], [Ko1]–[Ko3], [L]. Let us denote by

$$\mathcal{M}=\mathbb{C}^n \setminus \bigcup_{1\le i<j\le n}\{z_i=z_j\},$$

the configuration space of n distinct ordered points in $\mathbb{C}$. It is well–known that the fundamental group $\pi_1(\mathcal{M})$ of the space $\mathcal{M}$ coincides with the pure braid group P_n:

$$\pi_1(\mathcal{M})=P_n.$$

Let

$$w_{ij} = w_{ji} = \frac{1}{2\pi\sqrt{-1}} d\log(z_i - z_j)$$

be closed 1–form on $\mathcal{M}$. Then, $\{w_{ij} \mid 1 \le i < j \le n\}$ represents a basis for $H^1(\mathcal{M})$, and the relations among w_{ij}, $1 \le i < j \le n$, are generated by the Arnold relations

$$w_{ij} \wedge w_{jk} + w_{jk} \wedge w_{ik} + w_{ik} \wedge w_{ij} = 0$$

for $i < j < k$ (see, e.g. [A]).

Let $\{X_{ij} = X_{ji} \mid 1 \le i < j \le n\}$ be a set of non–commutative variables, and consider a formal connection

$$\Omega = \sum_{1 \le i < j \le n} w_{ij} X_{ij}. \qquad (2.3)$$

Lemma 2.3 (T. Kohno) *The connection Ω is integrable if and only if the following conditions are satisfied*

$$[X_{ij}, X_{ik} + X_{jk}] = [X_{ij} + X_{ik}, X_{jk}] = 0, \text{ for } i < j < k,$$
$$[X_{ij}, X_{kl}] = 0, \text{ for distinct } i, j, k, l.$$

Let us remind that $[a, b] = ab - ba$ is the usual commutator.

It is clear that these conditions are equivalent to the relations (2.1)–(2.2). A proof of Lemma 2.3 follows from the observations that $d\Omega = 0$, and the integrability condition for the connection Ω is equivalent to the following one: $\Omega \wedge \Omega = 0$. Notice that the connection (2.3) is the formal version of the so–called Knizhnik–Zamolodchikov connection in conformal field theory.

The relations (2.1) and (2.2) can be also interpreted [Ch2], [Dr2], [Ko2], as the self–consistency conditions

$$\frac{\partial A_j}{\partial z_i} - \frac{\partial A_i}{\partial z_j} = [A_i, A_j], \quad 1 \le i, j \le n,$$

of the Knizhnik–Zamolodchikov (system of) equation(s)

$$\frac{\partial \Phi}{\partial z_i} = k A_i \Phi, \quad 1 \le i \le n,$$

where $A_i = \sum_{j \ne i} \frac{X_{ij}}{z_i - z_j}$, and a function $\Phi := \Phi(z)$, $z = (z_1, \ldots, z_n)$, takes the values in a certain algebra (say over $\mathbb{C}((z))$) generated by the elements X_{ij}.

Below we formulate the basic properties of the quadratic algebra $\mathcal{B}_n$.

Theorem 2.4 ([Ko1], [Ko2]) *The Hilbert series of the algebra $\mathcal{B}_n$ is given by*

$$H(\mathcal{B}_n; t) = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}. \tag{2.4}$$

Corollary 2.5 ([Dr1], [L]) *A linear basis in the algebra $\mathcal{B}_n$ is given by the following set of noncommutative monomials*

$$Z_2 \cdot \ldots \cdot Z_n, \tag{2.5}$$

where

$$\begin{aligned} Z_2 &= \{1, X_{12}\}, \\ &\cdots \\ Z_k &= \{1\} \cup \{X_{i_1 k} \ldots X_{i_l k} \mid 1 \le i_j < k,\ 1 \le j \le l\}, \quad 2 \le k \le n. \end{aligned}$$

A monomial in X_{ij}'s is called to be in *normal form* if it belongs to the set (2.5).

Remark 2.6 Corollary 2.5 was stated without a proof in [Dr1]. The formula (2.4) was obtained in [Ko1] by constructing a free resolution of $\mathbb{C}$ as a trivial $\mathcal{B}_n$–module. Also, the formula (2.4) is a consequence of the fact that $\mathcal{B}_n$ is a semi-direct product of free associative algebras as shown in [Dr2]; see also [L]. Note, that the relations (2.1) can be interpreted as the horizontal version of the 4–term relations in the theory of Vassiliev link invariants.

Let us consider a completion $\widehat{\mathcal{B}}_n$ of the algebra $\mathcal{B}_n \otimes_{\mathbb{Z}} \mathbb{C}$ with respect to the powers of the ideal $\mathcal{I} = (X_{ij})$ generated by X_{ij}, $1 \le i < j \le n$. More precisely, let us denote by $\mathbb{C}\langle X_{ij}\rangle$ the ring of non–commutative formal power series with indeterminates X_{ij}, $1 \le i < j \le n$, and let J be an ideal generated by the relations (2.1) and (2.2). Then $\widehat{\mathcal{B}}_n = \mathbb{C}\langle X_{ij}\rangle / J$. It is well–known [Ko1], that there exists an isomorphism of complete Hopf algebras

$$\widehat{\mathbb{C}[P_n]} \cong \widehat{\mathcal{B}}_n,$$

where $\widehat{\mathbb{C}[P_n]}$ stands for the completion of the group ring of the pure braid group P_n with respect to the powers of the augmentation ideal.

3 Jucys–Murphy elements

Definition 3.1 *The Jucys–Murphy elements d_j, for $j = 2, \ldots, n$, in the quadratic algebra $\mathcal{B}_n$ are defined by*

$$d_j = \sum_{1 \le i < j} X_{ij}. \tag{3.1}$$

It is clear that d_i is a primitive element, i.e. $\Delta(d_i) = 1 \otimes d_i + d_i \otimes 1$.

Lemma 3.2 *Relations* $[X_{ik} + X_{jk}, X_{ij}] = 0$ *for* $i < j < k$, *together with commutativity relations (2.2), imply that the Jucys–Murphy elements* d_j *commute pairwise.*

Proof. For $j < l$,

$$[d_j, d_l] = \sum_{1 \le i < j,\ 1 \le k < l} [X_{ij}, X_{kl}] = \sum_{1 \le i < j < l} \{[X_{ij}, X_{il}] + [X_{ij}, X_{jl}]\} = 0.$$

■

Let us assume that additionally to the relations (2.1) and (2.2) the following relations are satisfied

$$X_{ij}^2 = 1, \quad 1 \le i < j \le n. \tag{3.2}$$

Then a map $(i < j)$

$$X_{ij} \to (i, j),$$

where $(i, j) \in S_n$ is the transposition that interchanges i and j and fixes each $k \neq i, j$, defines a representation of the relations (2.1), (2.2) and (3.2). In other words, there exists a homomorphism p from the braid algebra $\mathcal{B}_n$ to the group ring of the symmetric group, $p: \ \mathcal{B}_n \to \mathbb{Z}[S_n]$, such that $p(X_{ij}) = (i, j)$. Under this homomorphism p the element d_j maps to the Jucys–Murphy element in the group ring of the symmetric group S_n (see, e.g. [J], [Mu], [Ra]). A proof follows from the following relations between the transpositions in the symmetric group S_n: if $1 \le i < j < k \le n$, then

$$(ij)(ik) = (jk)(ij) = (ik)(jk),$$

$$(ij)(jk) = (ik)(ij) = (jk)(ik).$$

Theorem 3.3 *Let* $\mathcal{K}_n$ *be a commutative subalgebra of* $\mathcal{B}_n$ *generated by the Jucys–Murphy elements* d_j, $j = 2, \ldots, n$. *Then a map* $d_j \mapsto x_{j-1}$ *defines an isomorphism*

$$\mathcal{K}_n \cong \mathbb{Z}[x_1, \ldots, x_{n-1}].$$

There exists a multiplicative analog, denoted by D_k, of the Jucys–Murphy element d_k. Our construction of the elements D_k follows to [Ra].

Let g_i, $1 \le i \le n-1$, be the standard generators of the braid group B_n. Thus, the generators g_i satisfy the following relations

$$g_i g_j = g_j g_i, \quad \text{if} \quad |i - j| \ge 2, \tag{3.3}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad 1 \le i \le n-2. \tag{3.4}$$

Now let us define the multiplicative Jucys–Murphy elements D_k:

Definition 3.4 (cf. [Ra], (3.16)) *The multiplicative Jucys–Murphy elements D_k are defined by the following formulae*

$$D_2 = g_1^2; \ \ D_k = g_{k-1}g_{k-2}\cdots g_2 g_1^2 g_2 \cdots g_{k-2}g_{k-1}, \ \ 3 \le k \le n. \tag{3.5}$$

Lemma 3.5 *The elements D_k commute pairwise.*

Proof. First of all,

$$D_2D_3 = g_1^2 g_2 g_1^2 g_2 = g_1 g_2 g_1 g_2 g_1 g_2 = g_2 g_1^2 g_2 g_1^2 = D_3 D_2.$$

Now, using induction, we have ($3 \le k \le n-1$)

$$\begin{aligned} D_k D_{k+1} &= g_{k-1}D_{k-1}g_{k-1}g_k g_{k-1}D_{k-1}g_{k-1}g_k = g_{k-1}g_k D_{k-1}g_{k-1}D_{k-1}g_k g_{k-1}g_k \\ &= g_{k-1}g_k D_{k-1}g_{k-1}D_{k-1}g_{k-1}g_k g_{k-1} = g_{k-1}g_k D_{k-1}D_k g_k g_{k-1} \\ &= g_{k-1}g_k D_k D_{k-1}g_k g_{k-1} = D_{k+1}D_k. \end{aligned}$$

Finally, if $l-k \ge 2$, then

$$\begin{aligned} D_k D_l &= D_k g_{l-1}\cdots g_{k+1}D_{k+1}g_{k+1}\cdots g_{l-1} = g_{l-1}\cdots g_{k+1}D_k D_{k+1}g_{k+1}\cdots g_{l-1} \\ &= g_{l-1}\cdots g_{k+1}D_{k+1}D_k g_{k+1}\cdots g_{l-1} = D_l D_k. \end{aligned}$$

■

The elements D_k, $2 \le k \le n$, generate a commutative subgroup K_n in the braid group B_n. Now we are going to consider the following two problems: what is the infinitesimal deformation of the commutative subgroup K_n, and what is the quasi–classical limit

$$\mathbb{Z}[B_n] \to H_n(q) \to \mathbb{Z}[S_n] \tag{3.6}$$

of the multiplicative Jucys–Murphy elements D_k? In the quasi–classical limit (3.6) the group ring $\mathbb{Z}[B_n]$ of the braid group B_n is degenerated at first to the Iwahori–Hecke algebra $H_n(q)$, and then to the group ring $\mathbb{Z}[S_n]$ of the symmetric group S_n.

Proposition 3.6 *The quasi–classical limit (3.6) of the multiplicative Jucys–Murphy element $D_k \in B_n$ is equal to the Jucys–Murphy element $d_k \in \mathbb{Z}[S_n]$.*

Proof. Let us compute the quasi–classical limit (3.6) of the element D_k. The first step is to consider D_k as an element of the Iwahori–Hecke algebra $H_n(q)$. In other words, we have to add to (3.3) and (3.4) the new relations

$$g_i^2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})g_i + 1, \ \ 1 \le i \le n-1. \tag{3.7}$$

The next step is to consider the limit

$$\lim_{q\to 1} \frac{D_k - 1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

For this goal it is enough to compute $\frac{dD_k}{dq}|_{q=1}$. This can be done using the formula

$$\frac{dg_i}{dq} = \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})g_i + 2}{2q(q^{\frac{1}{2}} + q^{-\frac{1}{2}})}.$$

Thus,

$$\begin{aligned}
\frac{dD_k}{dq} &= \sum_{i=2}^{k-1} \left\{ g_{k-1} \cdots g_{i+1} \left(\frac{dg_i}{dq} \right) g_{i-1} g_2 g_1^2 g_2 \cdots g_{k-1} \right. \\
&+ \left. g_{k-1} \cdots g_2 g_1^2 g_2 \cdots g_{i-1} \left(\frac{dg_i}{dq} \right) g_{i+1} \cdots g_{k-1} \right\} \\
&+ g_{k-1} \cdots g_2 \left(\frac{d}{dq} g_1^2 \right) g_2 \cdots g_{k-1} \\
&\overset{q=1}{=} \sum_{i=2}^{k-1} \left\{ \frac{1}{2} s_{k-1} \cdots s_{i+1} s_i s_{i+1} \cdots s_{k-1} + \frac{1}{2} s_{k-1} \cdots s_{i+1} s_i s_{i+1} \cdots s_{k-1} \right\} \\
&+ s_{k-1} \cdots s_2 s_1 s_2 \cdots s_{k-1} \\
&= \sum_{i=1}^{k-1} (i,k) = d_k.
\end{aligned}$$

■

Finally, let us explain a connection between the multiplicative Jucys–Murphy elements D_k and the pure braid group P_n. More precisely, let us consider the infinitesimal deformations of the pure braid group P_n, and the multiplicative Jucys–Murphy elements D_k. To start, it is convenient to remind a few definitions.

By definition, the pure braid group P_n is a kernel of the natural homomorphism

$$B_n \to S_n, \quad g_i \mapsto s_i = (i, i+1),$$

where $s_i = (i, i+1) \in S_n$, $1 \le i \le n-1$, denote the transposition that interchanges i and $i+1$, and fixes all other elements of $[1, n]$. We have an exact sequence:

$$1 \to P_n \to B_n \to S_n \to 1.$$

It is well–known (see, e.g. [B]), that P_n is generated by the following elements

$$g_{ij} = (g_{j-1}g_{j-2}\cdots g_{i+1})g_i^2(g_{j-1}g_{j-2}\cdots g_{i+1})^{-1},\ 1 \le i < j \le n,$$

subject the following relations:

$$g_{ij}g_{kl} = g_{kl}g_{ij},\ \text{if all } i,j,k,l \text{ are distinct}, \tag{3.8}$$
$$g_{ij}g_{ik}g_{jk} = g_{ik}g_{jk}g_{ij} = g_{jk}g_{ij}g_{ik},\ \text{if } 1 \le i < j < k \le n, \tag{3.9}$$
$$g_{ik}g_{jk}g_{jl}g_{ij} = g_{jk}g_{jl}g_{ij}g_{ik},\ \text{if } 1 \le i < j < k < l \le n. \tag{3.10}$$

One can show using only the relations (3.8)–(3.10) that the elements

$$D_k = g_{1,k}g_{2,k}\cdots g_{k-1,k} \in P_n,\ 2 \le k \le n,$$

are mutually commute.

Now let us consider the infinitesimal deformation $g_{ij} \mapsto 1 + \epsilon X_{ij}$, of the pure braid group P_n. It is easy to see that the coefficients of ϵ^2 on both sides of relations (3.8)–(3.10) coincide with the defining relations (2.1)–(2.2) for the braid algebra $\mathcal{B}_n$, and the element D_k transforms to the Jucys–Murphy elements d_k (more precisely, $D_k = 1 + \epsilon d_k + o(\epsilon^2)$).

4 Quadratic Algebra $\mathcal{G}_n$

In the recent paper of the author and S. Fomin [FK] the quadratic algebra $\mathcal{E}_n^t$ was introduced. This algebra is closely related to the theory of quantum Schubert polynomials [FK], and their multiparameter deformation [K]. In this Section we introduce the quadratic algebra $\mathcal{G}_n$, which is a further generation of the quadratic algebra $\mathcal{E}_n^t$.

Definition 4.1 *Define the algebra $\mathcal{G}_n$ (of type A_{n-1}) as the quadratic algebra (say, over $\mathbb{Z}$) with generators $[ij]$, $1 \le i < j \le n$, which satisfy the following relations*

$$\begin{aligned} &[ij][jk] = [jk][ik] + [ik][ij]\,, \\ &[jk][ij] = [ik][jk] + [ij][ik], \quad \textit{for } i < j < k\,; \end{aligned} \tag{4.1}$$

$$\begin{aligned} &[ij][kl] = [kl][ij], \\ &\textit{whenever } \{i,j\} \cap \{k,l\} = \phi\,, i < j,\ \textit{and } k < l\,. \end{aligned} \tag{4.2}$$

The quadratic algebra $\mathcal{E}_n^t$, [FK], is the quotient of the algebra $\mathcal{G}_n$ by the two–side ideal generated by the relations

$$[ij]^2 - t_{ij} = 0,$$

where the t_{ij}, for $1 \le i < j \le n$, are a set of commuting parameters.

From (2.1) follows that the generators $[ij]$ satisfy the classical Yang–Baxter equation (CYBE)

$$[[ij],[ik]+[jk]]+[[ik],[jk]]=0,\ i<j<k, \tag{4.3}$$

where the external brackets stand for the usual commutator: $[a,b]=ab-ba$.

Theorem 4.2 *The Hilbert series of the algebra $\mathcal{G}_n$ is given by*

$$H(\mathcal{G}_n;t)=\frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}. \tag{4.4}$$

Theorem 4.2 is a corollary of the following result which gives a description of an additive basis in the algebra $\mathcal{G}_n$:

Theorem 4.3 *A linear basis of the algebra $\mathcal{G}_n$ is given by the following set of non-commutative monomials*

$$Z_2\cdot Z_3\cdot\ldots\cdot Z_n, \tag{4.5}$$

where

$$\begin{aligned} Z_2 &= \{\phi,[12]\} \\ &\cdots \\ Z_k &= \{\phi\}\cup\{[i_1k]\cdots[i_lk]\mid 1\le i_j<k,\ 1\le j\le l\},\quad 2\le k\le n. \end{aligned}$$

A monomial in $[ij]$'s is called to be in *normal form*, if it belongs to the set (4.5).

A proof of Theorem 4.2 is similar to a proof of Corollary 2.5 given by X.-S. Lin, [L], Theorem 2.3. By this reason we omit a proof.

Remark 4.4 It is clear that $\quad H(Z_k;t)=\dfrac{1}{1-(k-1)t}$, and

$$H(\mathcal{G}_n;t)=\prod_{k=2}^{n}H(Z_k;t). \tag{4.6}$$

Corollary 4.5 *Let $\mathcal{G}'_n$ be a commutative version of the algebra $\mathcal{G}_n$, i.e. let us assume that the relations (4.2) are valid for all i,j,k,l. Then*

$$H(\mathcal{G}'_n;t)=(1-t)^{-\binom{n}{2}}.$$

5 Dunkl elements

The Dunkl elements θ_j, for $j = 1, \ldots, n$, in the quadratic algebra $\mathcal{G}_n$ are defined by

$$\theta_j = -\sum_{1 \le i < j} [ij] + \sum_{j < k \le n} [jk]. \tag{5.1}$$

For the quadratic algebra $\mathcal{E}_n^t$ the Dunkl elements (5.1) coincide with those introduced in [FK], Section 5. The following result is well–known (cf., e.g. [Ki], Theorem 1.4, or [FK], Lemma 5.1)

Lemma 5.1 *The classical Yang–Baxter equation (4.3), together with the commutativity relation (4.2), imply that the Dunkl elements θ_j commute pairwise.*

There exists an obvious relation between Dunkl's elements $\theta_1, \ldots, \theta_n$ in the quadratic algebra $\mathcal{G}_n$, namely, $\theta_1 + \cdots + \theta_n = 0$. The result below shows that in the algebra $\mathcal{G}_n$ the Dunkl elements $\theta_1, \ldots, \theta_{n-1}$ are algebraically independent (cf. Theorem 7.3).

Theorem 5.2 *Let $\mathcal{H}_n$ be a commutative subalgebra of $\mathcal{G}_n$ generated by the Dunkl elements θ_j, $j = 1, \ldots, n$. Then a map $\theta_i \to x_i$, $1 \le i \le n-1$, defines an isomorphism*

$$\mathcal{H}_n \cong \mathbb{Z}[x_1, \ldots, x_{n-1}],$$

Problem 5.3 *To describe the two–side ideals $\mathcal{R} \subset \mathcal{G}_n$ such that*

$$H\left(\mathcal{H}_n / \mathcal{R} \cap \mathcal{H}_n; t\right) = [n]! := \prod_{j=1}^{n} \frac{1 - t^j}{1 - t}. \tag{5.2}$$

The examples of two–side ideals in the algebra $\mathcal{G}_n$ with the property (5.2) will be given in Sections 7 and 8. These examples are related to the quantum cohomology ring of the flag variety [FK], and the multiparameter deformation of Schubert polynomials [K].

Example 5.4 Let us take $n = 3$. In the algebra $\mathcal{G}_3$ we have the following relations

$$\begin{aligned}
&\theta_1 + \theta_2 + \theta_3 = 0, \\
&\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3 + [12]^2 + [13]^2 + [23]^2 = 0, \\
&\theta_1\theta_2\theta_3 + [12]^2\theta_3 + \theta_1[23]^2 - [13]^2\theta_1 - \theta_3[13]^2 = 0, \\
&[\theta_2, [23]^2] = [\theta_1, [13]^2],
\end{aligned}$$

where $[a, b] = ab - ba$ is the usual commutator.

6 Dunkl and Jucys–Murphy elements

Let us start with the definition of the extended affine braid group $\widetilde{B}_n$.

Definition 6.1 *The extended affine braid group $\widetilde{B}_n$ is a group with generators*

$$g_0, g_1, \ldots, g_{n-1}, w$$

which satisfy the following relations

$$g_i g_j = g_j g_i, \quad |i-j| \geq 2, \; 0 \leq i, j \leq n-1, \tag{6.1}$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad 0 \leq i \leq n-1, \tag{6.2}$$
$$w g_i = g_{i-1} w, \quad 0 \leq i \leq n-1, \tag{6.3}$$

with indices understood as elements of $\mathbb{Z}/n\mathbb{Z}$.

It follows from (6.3) that w^n is a central element.

There exists a canonical homomorphism π from the extended affine braid group $\widetilde{B}_n$ to the classical braid group B_n. On the generators g_i and w the homomorphism π is given by the following rules

$$\pi(g_i) = g_i, \; 1 \leq i \leq n-1, \tag{6.4}$$

$$\pi(w) = g_{n-1} g_{n-2} \cdots g_2 g_1.$$

It follows from (6.4) that

$$\pi(g_0) = \pi(w) g_1 \pi(w)^{-1} = g_{n-1} g_{n-2} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-2}^{-1} g_{n-1}^{-1}.$$

Now we are going to define the Dunkl elements $Y_1, \ldots, Y_n$, and the dual Dunkl elements $Y_1^*, \ldots, Y_n^*$ in the extended affine braid group.

Definition 6.2 *For each $i = 1, \ldots, n$, we define (cf. [KN]; [Ch1]) the Dunkl and dual Dunkl elements Y_i and Y_i^* respectively, by the following formulae*

$$Y_i \;=\; g_i g_{i+1} \ldots g_{n-1} w g_1^{-1} \ldots g_{i-1}^{-1}, \tag{6.5}$$
$$Y_i^* \;=\; g_i^{-1} g_{i+1}^{-1} \ldots g_{n-1}^{-1} w g_1 \ldots g_{i-1}. \tag{6.6}$$

Note that $Y_1 = g_1 \ldots g_{n-1} w$, $Y_n = w g_1^{-1} \ldots g_{n-1}^{-1}$, $Y_1^* = w g_1 \ldots g_{n-1}$ and $Y_n^* = g_1^{-1} \ldots g_{n-1}^{-1} w$.

The Dunkl elements satisfy the following commutation relations with $g_1, \ldots, g_{n-1}$:

$$g_i Y_{i+1} g_i \;=\; Y_i, \qquad g_i Y_j = Y_j g_i \;\; (j \neq i, i+1), \tag{6.7}$$
$$g_i Y_i^* g_i \;=\; Y_{i+1}^*, \quad g_i Y_j^* = Y_j g_i \;\; (j \neq i, i+1). \tag{6.8}$$

It is clear from our construction, that the image in the braid group B_n of the dual Dunkl element Y_k^*, $2 \le k \le n$, under the canonical homomorphism $\pi : \widetilde{B}_n \to B_n$ coincides with the multiplicative Jucys–Murphy element D_k, namely

$$\begin{aligned}\pi(Y_k^*) &= g_k^{-1} g_{k+1}^{-1} \cdots g_{n-1}^{-1} \pi(w) g_1 \cdots g_{k-1} \\ &= g_{k-1} \cdots g_2 g_1^2 g_2 \cdots g_{k-1} \overset{(3.5)}{=} D_k, \quad 2 \le k \le n,\end{aligned}$$

and $\pi(Y_1^*) = 1$.

Lemma 6.3 *The Dunkl elements Y_k (resp. the dual Dunkl elements Y_k^*) commute pairwise.*

Let us illustrate the main idea of the proof of Lemma 6.3 on some example. Let us take $n = 5$ and prove that $Y_2^* Y_3^* = Y_3^* Y_2^*$. Indeed,

$$\begin{aligned}Y_2^* Y_3^* &= g_2^{-1} g_3^{-1} g_4^{-1} w g_1 g_3^{-1} g_4^{-1} w g_1 g_2 = g_2^{-1} g_3^{-1} g_4^{-1} g_0 g_2^{-1} g_3^{-1} g_4 g_0 w^2 \\ &= g_2^{-1} g_3^{-1} g_2^{-1} g_4 g_3^{-1} g_0 g_4 g_0 w^2 = g_3^{-1} g_2^{-1} g_3^{-1} g_4 g_3^{-1} g_4 g_0 g_4 w^2.\end{aligned}$$

Similarly,

$$Y_3^* Y_2^* = g_3^{-1} g_4^{-1} w g_1 g_3^{-1} g_4^{-1} w g_1 = g_3^{-1} g_2^{-1} g_4^{-1} g_3^{-1} g_0 g_4 w^2.$$

It is easy to see that in $\widetilde{B}_5$ we have

$$g_3^{-1} g_2^{-1} g_3^{-1} g_4 g_3^{-1} g_4 g_0 g_4 = g_3^{-1} g_2^{-1} g_4^{-1} g_3^{-1} g_0 g_4.$$

Now let us study the quasi–classical limit of the Dunkl element Y_k. The first step is to consider Y_k as an element of the extended affine Hecke algebra $H(\widetilde{W})$, [KN], Section 2; [Ch1], [Ch2]. In other words, we have to add to the relations (6.1)–(6.3) the new ones

$$g_i^2 = (t-1) g_i + t, \quad 0 \le i \le n \tag{6.9}$$

where t is a new parameter. The next step is to consider a representation of the extended affine Hecke algebra $H(\widetilde{W})$ in the algebra $\mathcal{D}_{q,x}[W]$ of q–difference operators with permutations, see, for example, [Ch1], [KN]. Follow [KN], we define the elements T_i, $i = 0, 1, \ldots, n-1$, in $\mathcal{D}_{q,x}[W]$ by

$$\begin{aligned}T_i &= t + \frac{x_{i+1} - t x_i}{x_{i+1} - x_i}(s_i - 1), \quad i = 1, \ldots, n-1, \\ T_0 &= t + \frac{x_1 - tq x_n}{x_1 - q x_n}(s_0 - 1).\end{aligned}$$

Here $s_1, \ldots, s_{n-1}$ are the standard generators of the symmetric group S_n, and $s_0 = w s_1 w^{-1}$, where $w = s_{n-1} s_{n-2} \cdots s_1 \tau_1$, and $\tau_1 = T_{q,x_1}$ is the q–shift operator:

$$(\tau_1 f)(x_1, \ldots, x_n) = f(q x_1, x_2, \ldots, x_n).$$

One can check that the elements T_i, $0 \le i \le n-1$, and w, satisfy the relations (6.1)–(6.3) and (6.9), and define a representation of the extended affine Hecke algebra $H(\widetilde{W})$ in the algebra $\mathcal{D}_{q,x}[W]$ of q–difference operators with permutations. In this representation, the Dunkl element Y_k (resp. Y_k^*), $1 \le k \le n$, up to a power of t coincides (see [Ch1], [Ch2], [KN]) with the Dunkl–Cherednik operator $\mathbf{Y}_k$:

$$\begin{aligned} \mathbf{Y}_k &= t^{-n+2k-1} T_k T_{k+1} \ldots T_{n-1} w t_1^{-1} \ldots T_{k-1}^{-1}, && (6.10) \\ \mathbf{Y}_k^* &= t^{n-2k+1} T_k^{-1} T_{k+1}^{-1} \ldots T_{n-1}^{-1} w T_1 \ldots T_{k-1}. && (6.11) \end{aligned}$$

In order to understand better the quasi–classical behavior of the Dunkl–Cherednik operator $\mathbf{Y}_k$, it is convenient to rewrite slightly the formulae (6.10)–(6.11). Namely, let us define ($1 \le i < j \le n$) the following operators acting on the ring of polynomials $\mathbb{Z}[t,t^{-1}][x_1,\ldots,x_n]$

$$\overline{T}_{ij} = 1 + \frac{(1-t^{-1})x_j}{x_i - x_j}(1 - s_{ij}) = t^{-1} T_{ij} s_{ij},$$

where s_{ij} is the exchange operator for variables x_i, x_j. Then

$$\mathbf{Y}_i = \overline{T}_{i,i+1} \overline{T}_{i,i+2} \ldots \overline{T}_{i,n} \tau_i \overline{T}_{1,i}^{-1} \ldots \overline{T}_{i-1,i}^{-1}, \tag{6.12}$$

where $\tau_i = s_i s_{i+1} \ldots s_{n-1} w s_1 \ldots s_{i-1}$ is the q–shift operator: $\tau_i = \tau_{x_i} = T_{q,x_i}$. Finally, in the quasi–classical limit $q \to 1$ with rescaling $t = q^\beta$, we obtain from (6.12) that

$$\mathcal{D}_j := \lim_{q \to 1} \frac{1 - \mathbf{Y}_j}{1-q} = x_j \frac{\partial}{\partial x_j} + \beta \sum_{i<j} x_j \partial_{ij} - \beta \sum_{k>j} \partial_{jk} x_k. \tag{6.13}$$

Note that these Dunkl operators $\mathcal{D}_k$ commute with each other, i.e. $[\mathcal{D}_i, \mathcal{D}_j] = 0$.

More generally, let us define the affine extension of the algebra $\mathcal{G}_n$:

Definition 6.4 *Define the algebra $\widetilde{\mathcal{G}}_n$ as the algebra (say over $\mathbb{Z}[q,q^{-1}]$)with generators*

$$[ij], \quad 0 \le i < j \le n; \quad x_i, \; 1 \le i \le n, \quad \text{and } w,$$

which satisfy the relations (4.1) and (4.2) for $i,j,k,l \in [1,n]$, and the following additional relations

$$\begin{aligned} x_i x_j &= x_j x_i, \quad 1 \le i,j \le n, && (6.14) \\ x_i [ab] &= [ab] x_i, \ \text{ if } i \ne a,b, && (6.15) \\ x_j [ij] &= [ij] x_i + 1, \quad x_i [ij] = [ij] x_j - 1 \quad 1 \le i < j \le n, && (6.16) \\ w x_i &= q^{\delta_{1,i}} x_{i-1} w, && (6.17) \\ w [ij] &= q^{-\delta_{i,0}} [i-1, j-1] w, && (6.18) \end{aligned}$$

with indices understood as elements of $\mathbb{Z}/n\mathbb{Z}$.

It is clear from (6.16), that

$$[x_i + x_j, [ij]] = 0. \tag{6.19}$$

We define the Dunkl elements $\widetilde{\theta}_j$ in the algebra $\widetilde{\mathcal{G}}_n$ by the following rule

$$\widetilde{\theta}_j = x_j + \theta_j, \quad 1 \le j \le n. \tag{6.20}$$

The following result is well-known, see, e.g. [Ki]:

Lemma 6.5 *The classical Yang–Baxter equation (4.3), together with commutativity relation (4.2) and relation (6.19), imply that the Dunkl elements $\widetilde{\theta}_j = x_j + \theta_j$ commute pairwise.*

Proof. If $i < j$, then

$$[\widetilde{\theta}_i, \widetilde{\theta}_j] = [x_i, x_j] + [x_i, \theta_j] + [\theta_i, x_j] + [\theta_i, \theta_j] = [x_i + x_j, [ij]] = 0.$$

■

Let us introduce the algebra $\widetilde{\mathcal{G}}_n^0$ which is the quotient of the algebra $\widetilde{\mathcal{G}}_n$ by the two–side ideal generated by the relation

$$[ij]^2 = 0, \quad 1 \le i < j \le n. \tag{6.21}$$

Definition 6.6 *For $1 \le i < j \le n$ let us define an element $T_{ij} \in \widetilde{\mathcal{G}}_n^0$ by the following formula (cf. [KN])*

$$T_{ij} = t - (x_j - t x_i)[ij]. \tag{6.22}$$

It follows from (6.16) and (6.21), that

$$T_{ij}^2 = (t-1)T_{ij} + t.$$

Our next goal is to check that the elements $\widetilde{T}_i := T_{i,i+1}$, $1 \le i \le n-1$, satisfy the Coxeter relations.

Proposition 6.7 *We have the following relation in the algebra $\widetilde{\mathcal{G}}_n^0$:*

$$T_{ab}T_{bc}T_{ab} = T_{bc}T_{ab}T_{bc}, \quad a < b < c. \tag{6.23}$$

Proof. Direct computation based on (6.16) and the following relation in the algebra $\widetilde{\mathcal{G}}_n^0$:

$$[ab][bc][ab] = [bc][ab][bc], \quad 1 \le a < b < c \le n.$$

■

As a corollary, we see that the elements w, $\widetilde{T}_0 = w\widetilde{T}_1 w^{-1}$, $\widetilde{T}_1, \ldots, \widetilde{T}_{n-1}$ generate the representation of the extended affine Hecke algebra $H(\widetilde{W})$. In this representation

the quasi–classical limit $q \to 1$ with rescaling $t = q^{\beta}$ of the Dunkl element $t^{-n+2j-1}Y_j$ is equal to

$$x_j \frac{\partial}{\partial x_j} + \beta \sum_{i<j} x_j[ij] - \beta \sum_{k>j} [jk] x_k. \tag{6.24}$$

The element (6.24) does not belong to the algebra $\widetilde{\mathcal{G}}_n$, but its further extension $\widetilde{\widetilde{\mathcal{G}}}_n$ which is an analog of the double affine Hecke algebra, introduced by I. Cherednik. It is an interesting problem to understand some connections between the Schubert calculus and the Dunkl–Cherednik operators.

Conjecture 6.8 (Nonnegativity conjecture, cf. [FK], Conjecture 8.1) *For any $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w$ evaluated at the Dunkl elements $\widetilde{\theta}_1, \ldots, \widetilde{\theta}_n$ belongs to the positive cone $\widetilde{\mathcal{G}}_n^{+}$, where $\widetilde{\mathcal{G}}_n^{+}$ is the cone of all nonnegative integer linear combinations of all (noncommutative in general) monomials in the generators x_i, $1 \le i \le n$, and $[ij]$, $1 \le i < j \le n$, of the algebra $\widetilde{\mathcal{G}}_n^{0}$.*

7 Quadratic algebra $\mathcal{E}_n^t$

In this and the next sections we study some interesting quotients of the the quadratic algebra $\mathcal{G}_n$.

Definition 7.1 (cf. [FK], Section 15) *Define the algebra $\mathcal{E}_n^t$ as the algebra over $\mathbb{Z}$ with generators $[ij]$, $1 \le i < j \le n$, which satisfy the relations (4.1), (4.2) and the additional relations*

$$([ij],[kl]^2) = 0, \quad for\ all \quad i,j,k,l. \tag{7.1}$$

The relations (7.1) mean that the squares $[kl]^2$ belong to the center of $\mathcal{E}_n^t$. Let us put $t_{ij} = [ij]^2$, and consider the ring of polynomials $\mathbb{Z}[t] := \mathbb{Z}[\ t_{ij},\ 1 \le i < j \le n]$.

Conjecture 7.2 ([FK]) *The algebra $\mathcal{E}_n^t$ is a finite dimensional module over the ring of polynomials $\mathbb{Z}[t]$.*

Theorem 7.3 ([FK], Conjecture 15.1; [P]) *Let $\mathcal{H}_n^t$ be a commutative subalgebra of $\mathcal{E}_n^t$ generated by the Dunkl elements $\theta_1, \ldots, \theta_n$. Then*

$$\mathcal{H}_n^t \cong \mathbb{Z}[t][\theta_1, \ldots, \theta_n]/J,$$

where the ideal J is generated by the generalized elementary functions

$$e_m(X_n|t) := \sum_l \sum_{\substack{1 \le i_1 < \cdots < i_l \le n \\ j_1 > i_1, \ldots, j_l > i_l}} e_{m-2l}(X_{\overline{I \cap J}}) \prod_{k=1}^{l} t_{i_k j_k}, \tag{7.2}$$

where $i_1, \ldots, i_l, j_1, \ldots, j_l$ should be distinct elements of the set $\{1, \ldots, n\}$, and $X_{\overline{I \cap J}}$ denotes set of variables x_a, for which the subscript a is neither one of the I_k nor one of the J_k.

One can check that the relations (7.1) are equivalent to the following ones

$$(\theta_i, [kl]^2) = 0, \quad \text{for all} \quad i,k,l.$$

The relations (7.1) in the algebra $\mathcal{E}_n^t$ imply some nontrivial relations between monomials in the space $Z := Z_2 \cdot Z_3 \cdot \ldots \cdot Z_n$. We will describe part of these relations in Proposition 7.4 below, but before that, let us introduce some additional notations. For given $n \ge 3$ and k, $1 \le k < n$, let us denote by $\Lambda_{n,k}$ a set of all sequences of integer numbers $A = (a_1, \ldots, a_k)$, such that $a_1 + \cdots + a_k = n - 2$, and $a_i \in \mathbb{Z}_{>0}$.

For each sequence A from the set $\Lambda_{n,k}$ let us define a monomial $[A]$ in the space Z as the following ordered product

$$[A] = \prod_{j=1}^{k} \prod_{l=1}^{a_j} [n-k+l-2-\sum_{s=1}^{j}(a_s-1),\ n-k+j-1].$$

For example, for $n=7$ and $A=(2,1,2)$ we have

$$[A] = [24][34][25][16][26].$$

Proposition 7.4

$$F_n := \sum_{i=1}^{n-1} [i,n][i+1,n]\cdots[n-1,n][1,n][2,n]\cdots[i,n] \tag{7.3}$$

$$= \sum_{k=2}^{n-1} (-1)^{n-k-1}(t_{kn}-t_{k-1,n}) \sum_{A\in\Lambda_{n,n-k}} [A].$$

Proof. By induction, using the following formula

$$\begin{aligned}&\Phi_{n+1}\cdot[12]-[12]\cdot\Phi_{n+1} = F_{n+1}, \quad \text{where}\\ &\Phi_n := [2,n]\cdots[n-1,n][2,n]+\cdots+[n-1,n][2,n]\cdots[n-2,n][n-1,n].\end{aligned}$$

∎

Let us make the following simple but useful observation. The action of the symmetric group, namely, $w([i_1j_1]\cdots[i_kj_k]) = [w(i_1)w(j_1)]\cdots[w(i_k)w(j_k)]$, $w\in S_n$, transforms every specific identity in $\mathcal{G}_n$ (resp. $\mathcal{E}_n^t$, $\mathcal{E}_n^0$) into an orbit of identities. As an example, the identity (in $\mathcal{E}_n^t$)

$$[12][13][14][12]+[13][14][12][13]+[14][12][13][14] = (t_{34}-t_{24})[13][23]-(t_{24}-t_{14})[12][13]$$

(a special case of Proposition 7.4) immediately implies a more general identity

$$\begin{aligned}&[ab][ac][ad][ab]+[ac][ad][ab][ac]+[ad][ab][ac][ad]\\ &=([cd]^2-[bd]^2)[ac][bc]-([bd]^2-[ad]^2)[ab][ac].\end{aligned}$$

Proposition below describes some interesting relations in the quadratic algebra $\mathcal{E}_n^t$. They can be interpreted as the Pieri rules for the classical and quantum Schubert polynomials, [FK], [P], and for the multiparameter deformation of Schubert polynomials [K], [P].

Proposition 7.5 ([P]) *Let A be a subset in $[1,\ldots,n]$, and let us put $\theta_A = \prod_{i \in A} \theta_i$. Then in the algebra $\mathcal{E}_n^t$ we have the following relation*

$$e_m(\theta_A|t) = \sum [i_1 j_1] \cdots [i_m j_m],$$

where the summation is taken over all sequences $I = \{i_1,\ldots,i_m\}$, $J = \{j_1,\ldots,j_m\}$ such that

i) $I \subset A$, $J \subset [1,n] \setminus A$,

ii) $i_1,\ldots,i_m$ *are all distinct,*

iii) $1 \le j_1 \le j_2 \le \cdots \le j_m \le n$.

Problem 7.6 *Find a nontrivial faithful representation of the quadratic algebra $\mathcal{E}_n^t$.*

8 Quadratic algebra $\mathcal{E}_n^0$

In this Section we study the properties of the algebra $\mathcal{E}_n^t$ when all parameters t are equal to zero. Hence, in this Section we assume that

$$[ij]^2 = 0, \quad 1 \le i < j \le n. \tag{8.1}$$

This algebra was introduced and studied at first in [FK]. It follows from Theorem 4.3 that any element of $\mathcal{E}_n^0$ can be presented as $\mathbb{Z}$-linear combination of monomials z of the following form

$$z = z_1 z_2 \cdots z_n, \tag{8.2}$$

where $z_i \in Z_i$, and $(2 \le k \le n)$

$$Z_k := Z_{n,k} = \{\phi\} \cup \{[i_1 k] \cdots [i_l k] \mid 1 \le i_l < k,\ 1 \le j \le l\}. \tag{8.3}$$

In the algebra $\mathcal{G}_n$ the monomials in the set Z_k are linearly independent (Theorem 4.3). However, in the quotient algebra $\mathcal{E}_n^0$ some new relations between the monomials in Z_k are appeared. The Proposition 8.1 below is a special case of Proposition 7.4 and gives the relations in subalgebra $Z_m \subset \mathcal{E}_n^0$, but not only in the space $Z = Z_2 \cdot Z_3 \cdot \ldots \cdot Z_n$.

Proposition 8.1 (cf. [FK]) *Let $a_1,\ldots,a_m$ be a sequence of pairwise distinct integer numbers, $1 \le a_i < n$, $m \le n$. Then*

$$a_1 a_2 \cdots a_m a_1 + a_2 a_3 \cdots a_1 a_2 + \cdots + a_m a_1 a_2 \cdots a_{m-1} a_m = 0. \tag{8.4}$$

Here we used an abbreviation $a_j = [a_j n]$.

Proof. It is enough to prove that

$$F_n := \sum_{i=1}^{n-1} [i,n][i+1,n]\cdots[n-1,n][1,n][2,n]\cdots[i,n] = 0. \tag{8.5}$$

Let us prove (8.5) by induction. Thus we may assume that $F_k = 0$ for $k \le n$. Let us consider the following element in Z_{n+1}:

$$\Phi_n = [2,n+1]\cdots[n,n+1][2,n+1]+\cdots+[n,n+1][2,n+1]\cdots[n-1,n+1][n,n+1].$$

By induction assumption, we have $\Phi_n = 0$. On the other hand, it is easy to check that

$$\Phi_n \cdot [12] - [12]\cdot \Phi_n = F_{n+1}. \tag{8.6}$$

The last equality shows that $F_{n+1} = 0$. ∎

Example 8.2 If $m = 3$, the relation (8.4) has the following form ($a < b < n$)

$$aba + bab = 0, \quad \text{in the algebra } \ \mathcal{E}_n^0 \tag{8.7}$$

Similarly, if a, b, c are distinct, $< n$, then

$$abca + bcab + cabc = 0, \quad \text{in the algebra } \ \mathcal{E}_n^0 \tag{8.8}$$

We don't know how to describe all relations between monomials from the space Z_k for fixed k, but we can show that if $k \le 5$, then all relations between monomials in Z_k follows from (8.4). This observation allows us to compute the dimensions of the homogeneous components $\mathcal{E}_{n,k}^0$ for $1 \le k \le 5$, and also $\dim \mathcal{E}_{n,6}^0$. We summarize the results in the following

Proposition 8.3

- $\dim \mathcal{E}_{n,1}^0 = \binom{n}{2}$,
- $\dim \mathcal{E}_{n,2}^0 = 3\binom{n}{4} + 4\binom{n}{3}$,
- $\dim \mathcal{E}_{n,3}^0 = 15\binom{n}{6} + 40\binom{n}{5} + 30\binom{n}{4} + 3\binom{n}{3}$,
- $\dim \mathcal{E}_{n,4}^0 = 105\binom{n}{8} + 420\binom{n}{7} + 610\binom{n}{6} + 366\binom{n}{5}$
  $+67\binom{n}{4} + \binom{n}{3}$,

- $\dim \mathcal{E}^0_{n,5} = 945\binom{n}{10} + 5040\binom{n}{9} + 10780\binom{n}{8} + 11571\binom{n}{7}$
$+6285\binom{n}{6} + 1480\binom{n}{5} + 96\binom{n}{4}$,
- $\dim \mathcal{E}^0_{n,6} = 10395\binom{n}{12} + 69300\binom{n}{11} + 195300\binom{n}{10} + 299908\binom{n}{9}$
$+268674\binom{n}{8} + 138545\binom{n}{7} + 37456\binom{n}{6} + 4231\binom{n}{5}$
$+106\binom{n}{4}$.

Conjecture 8.4

$$\dim \mathcal{E}^0_{n,k} = (2k-1)!!\binom{n}{2k} + (2k-1)!!\frac{4(k-1)}{3}\binom{n}{2k-1} \tag{8.9}$$
$$+ (2k-3)!!\frac{(k-1)(k-2)(16k-3)}{9}\binom{n}{2k-2} + \cdots.$$

Proof of Proposition 8.3. First of all, let us compute $\dim Z_{n,k}$ for small values of k. It is clear that $\dim Z_{n,1} = n-1$ and $\dim Z_{n,2} = (n-1)(n-2)$, because there are no nontrivial relations of degree < 3 in Z_n. In degree 3 we have to take into account the relations (8.7). Thus,

$$\dim Z_{n,3} = (n-2)\dim Z_{n,2} - \binom{n-1}{2} = \frac{(n-1)(n-2)(2n-5)}{2}.$$

Similarly, in degree 4 we have to consider both the relations (8.7) and (8.8). Thus,

$$\dim Z_{n,4} = (n-2)\dim Z_{n,3} - (n-2)\binom{n-1}{2} - 2\binom{n-1}{3}$$
$$= \frac{(n-1)(n-2)(n-3)(3n-7)}{3} = 24\binom{n-1}{4} + 10\binom{n-1}{3}.$$

Using the same method, we can compute

$$\dim Z_{n,5} = 120\binom{n-1}{5} + 86\binom{n-1}{4} + 9\binom{n-1}{3}.$$

When $k \geq 6$, some additional to (8.4) relations between monomials in Z_k will appear. We describe such new relations in degree 6.

Lemma 8.5 *Let a, b, c, d be a set of distinct integer numbers. Then*

$$abcdca + bcdcab + cabadc + dcabad + cdcaba \tag{8.10}$$
$$= abacdc + acdcba + bacdcb + cdabac + dabacd.$$

where $a := [an], \;\; n \geq 5$.

Proof of Lemma 8.5. Let us consider the difference

$$[ac]F_4(a,d,c,b) - F_4(a,b,c,d)[ac], \tag{8.11}$$

where

$$\begin{aligned} F_4(a,b,c,d) &= [an][bn][cn][dn][an] + [bn][cn][dn][an][bn] \\ &+ [cn][dn][an][bn][cn] + [dn][an][bn][cn][dn]. \end{aligned}$$

First of all, the difference (8.11) is equal to zero. On the other hand, using the relations in the algebra $\mathcal{E}_n^0$, one can transform the expression (8.11) to the difference between the left and the right hand sides of (8.10). ■

Using Lemma 8.5, we can compute

$$\dim Z_{n,6} = 720 \begin{pmatrix} n-1 \\ 6 \end{pmatrix} + 756 \begin{pmatrix} n-1 \\ 5 \end{pmatrix} + 187 \begin{pmatrix} n-1 \\ 4 \end{pmatrix} + 6 \begin{pmatrix} n-1 \\ 3 \end{pmatrix}.$$

Finally, in order to compute the dimensions of homogeneous components $\mathcal{E}_{n,k}^0$, $1 \leq k \leq 6$, we use an observation that monomials from the set $Z_2 \cdot Z_3 \cdot Z_4 \cdot Z_5 \cdot Z_6$ form a basis in $\displaystyle\bigoplus_{k=0}^{6} \mathcal{E}_{n,k}^0$. The proof of Proposition 8.3 is finished. ■

Conjecture 8.6 (Factorization property)

$$H(\mathcal{E}_n^0; t) = H(Z_{n,2}; t) H(Z_{n,3}; t) \cdots H(Z_{n,n}; t). \tag{8.12}$$

Example 8.7 If $n = 2$, then $Z_2 = \{\phi\} \cup \{12\}$, and $H(Z_2; t) = 1 + t$.

If $n = 3$, then $Z_3 = \{\phi\} \cup \{13,\ 23,\ 13 \cdot 23,\ 23 \cdot 13,\ 13 \cdot 23 \cdot 13\}$, and

$$H(Z_3; t) = 1 + 2t + 2t^2 + t^3.$$

Now, let us consider $n = 4$. Then ($a := [an]$)

$$\begin{aligned} Z_{4,1} &= \{1,2,3\}, \\ Z_{4,2} &= \{12, 13, 21, 31, 23, 32\}, \\ Z_{4,3} &= \{121, 123, 131, 132, 213, 312, 231, 232, 321\}, \\ Z_{4,4} &= \{1213, 1231, 1232, 1312, 1321, 2131, 2132, 3121, 2312, 2321\}, \\ Z_{4,5} &= \{12131, 12132, 12312, 12321, 13121, 21312, 21321, 31213, 23121\}, \\ Z_{4,6} &= \{121312, 121321, 123121, 131213, 213121, 231213\}, \\ Z_{4,7} &= \{1213121, 1231213, 2131213\}, \\ Z_{4,8} &= \{12131213\}. \end{aligned}$$

Hence, $H(Z_4;t) = (1+t)^2(1+t+t^2)(1+t^2)^2$, and

$$H(\mathcal{E}_4^0;t) = (1+t)^4(1+t^2)^2(1+t+t^2)^2.$$

Formula for $H(\mathcal{E}_4^0;t)$ was appeared in [FK] and confirmed by J.-E. Ross [Ro], who also computed the Hilbert polynomial $H(\mathcal{E}_5^0;t)$.

9 Quadratic algebra $\mathcal{B}_n^t$

In this Section we define a certain quotient $\mathcal{B}_n^t$ of the braid algebra $\mathcal{B}_n$, and formulate a conjecture that the algebras $\mathcal{B}_n^t$ and $\mathcal{E}_n^t$ have the same Hilbert polynomials.

Definition 9.1 *define the algebra $\mathcal{B}_n^t$ as the algebra over $\mathbb{Q}$ with generators X_{ij}, $1 \le i < j \le n$, which satisfy the relations (2.1), (2.2) and the additional relations*

$$[ij]^2 = t_{ij}, \tag{9.1}$$

where the t_{ij} for $1 \le i < j \le n$, are a set of commuting parameters.

In the case when all parameters t_{ij} are equal to zero, we denote the algebra $\mathcal{B}_n^t$ by $\mathcal{B}_n^0$.

Conjecture 9.2 *The algebra $\mathcal{B}_n^t$ is a finite dimensional module over the ring of polynomials $\mathbb{Q}[t] := \mathbb{Q}[t_{ij}, 1 \le i < j \le n]$.*

Problem 9.3 *To find all relations among the Jucys–Murphy elements d_j, $2 \le j \le n$, and find the Hilbert polynomial of commutative subalgebra $\mathcal{K}_n^0$ of the algebra $\mathcal{B}_n^0$ generated by the Jucys–Murphy elements d_j, $2 \le j \le n$.*

Example 9.4 Let us consider the algebra $\mathcal{B}_3^0$. This is an algebra over $\mathbb{Q}$ with generators X_{12}, X_{13}, X_{23} subject the following quadratic relations

$$X_{12}^2 = X_{13}^2 = X_{23}^2 = 0, \tag{9.2}$$

$$X_{13}X_{12} = X_{12}X_{13} - X_{13}X_{23} + X_{23}X_{13}, \tag{9.3}$$

$$X_{23}X_{12} = X_{12}X_{23} - X_{23}X_{13} + X_{13}X_{23}. \tag{9.4}$$

Using these relations, we can find the new relations in the algebra $\mathcal{B}_3^0$, namely,

$$X_{23}X_{13}X_{23} - X_{13}X_{23}X_{13} + X_{12}X_{13}X_{23} - X_{12}X_{23}X_{13} = 0, \tag{9.5}$$

$$X_{23}X_{13}X_{23}X_{13} + X_{12}X_{13}X_{23}X_{13} = 0, \tag{9.6}$$

$$X_{13}X_{23}X_{13}X_{23} + X_{12}X_{23}X_{13}X_{23} = 0, \tag{9.7}$$

$$X_{12}X_{23}X_{13}X_{23} - X_{12}X_{13}X_{23}X_{13} = 0. \tag{9.8}$$

Remark 9.5 To be more precise, from the relations (9.2)–(9.4) follow only that

$$2 \cdot \mathrm{LHS}(9.5) = 0$$

We expect that the $\mathbb{Z}$–form of the quadratic algebra $\mathcal{B}_n^0$ has only 2–torsion.

It is easy to check from the relations (9.2)–(9.8) that the Hilbert polynomial of the algebra $\mathcal{B}_3^0$ is equal to

$$H(\mathcal{B}_3^0;t) = 1 + 3t + 4t^2 + 3t^3 + t^4 = (1+t)^2(1+t+t^2) = H(\mathcal{E}_3^0;t).$$

Now, let us consider the commutative subalgebra $\mathcal{K}_3^0 := \mathbb{Q}[d_2, d_3]$, where $d_2 = X_{12}$ and $d_3 = X_{13} + X_{23}$. The Jucys–Murphy elements d_2 and d_3 satisfy the following relations

$$d_2^2 = 0, \quad d_2 d_3 = d_3 d_2, \quad (d_2 + d_3) d_3^3 = 0,$$

and the elements $1, d_2, d_3, d_2d_3, d_3^2, d_2d_3^2, d_3^3, d_2d_3^3$ form a linear basis in the algebra $\mathcal{K}_3^0$. Thus we have

$$\dim \mathcal{K}_3^0 = 8, \quad H(\mathcal{K}_3^0;t) = (1+t)^2(1+t^2).$$

For general n we expect that

$$H(\mathcal{K}_n^0;t) = \prod_{k=1}^{n-1} \frac{1-t^{2k}}{1-t} = H(W(B_{n-1});t), \tag{9.9}$$

where $W(B_{n-1})$ is the Weyl group of type B_{n-1}.

Conjecture 9.6 *The quadratic algebras $\mathcal{B}_n^0$ and $\mathcal{E}_n^0$ have the same Hilbert polynomials*

$$H(\mathcal{B}_n^0;t) = H(\mathcal{E}_n^0;t). \tag{9.10}$$

We can check (9.10) for $n = 3$ (Example 9.4) and $n = 4$. In order to prove the equality (9.10) for $n = 4$, we use the same method as in Example 8.7. But, instead of identity (8.8) in the algebra $\mathcal{E}_n^0$, we have to use the following 14–terms relations in the algebra $\mathcal{B}_n^0$, $n \geq 4$:

$$\begin{aligned} &X_{23}(X_{14}X_{24}X_{34} - X_{24}X_{14}X_{34} - X_{34}X_{14}X_{24} + X_{34}X_{24}X_{14}) \\ &+(X_{13}X_{23} - X_{23}X_{13})(X_{24}X_{34} - X_{34}X_{24}) - X_{14}X_{24}X_{34}X_{24} \\ &+X_{24}X_{34}X_{24}X_{14} + X_{24}X_{14}X_{34}X_{24} - X_{24}X_{34}X_{14}X_{24} \\ &+X_{34}X_{14}X_{24}X_{34} - X_{34}X_{24}X_{14}X_{34} = 0, \end{aligned} \tag{9.11}$$

$$\begin{aligned} &X_{12}(X_{14}X_{34}X_{24} - X_{24}X_{14}X_{34} - X_{34}X_{14}X_{24} + X_{24}X_{34}X_{14}) \\ &+(X_{13}X_{23} - X_{23}X_{13})(X_{14}X_{24} - X_{24}X_{14}) + X_{14}X_{24}X_{34}X_{14} \\ &-X_{14}X_{34}X_{24}X_{14} - X_{14}X_{24}X_{14}X_{34} + X_{24}X_{14}X_{34}X_{24} \\ &-X_{24}X_{34}X_{14}X_{24} + X_{34}X_{14}X_{24}X_{14} = 0. \end{aligned} \tag{9.12}$$

It is also natural to ask whether or not the quadratic algebra $\mathcal{B}_n^0$ has the Koszul property (see, e.g. [Ma] and references therein).

10 Quadratic algebra $\mathcal{A}_n^t$

In this section we consider the quadratic algebra $\mathcal{A}_n^t$ (denoted by $\mathcal{E}C_n$ in [FK], Section 4.3) which is the commutative quotient of the algebra $\mathcal{E}_n^t$.

Definition 10.1 *Define the algebra $\mathcal{A}_n^t$ (of type A_{n-1}) as the quadratic algebra over $\mathbb{Z}$ with generators $[ij]$, $1 \le i < j \le n$, which satisfy the following relations*

$$[ij][kl] = [kl][ij], \quad \text{for all } i, j, k, l; \tag{10.1}$$

$$[ij][jk] = [ik][ij] + [ik][jk], \quad i < j < k; \tag{10.2}$$

$$[ij]^2 = t_{ij}. \tag{10.3}$$

Let us denote by $\mathcal{A}_n^0$ the specialization $t_{ij} = 0$, $1 \le i < j \le n$, of the algebra $\mathcal{A}_n^t$.

Theorem 10.2 (cf. [A]) *The Hilbert polynomial of the algebra $\mathcal{A}_n^0$ is given by*

$$H(\mathcal{A}_n^0; t) = (1 + t)(1 + 2t) \cdots (1 + (n - 1)t). \tag{10.4}$$

Corollary 10.3 (cf. [A]) *An additive basis of the algebra $\mathcal{A}_n^t$ is given by the following set of (commutative) monomials*

$$[j_1 k_1] \dots [j_l k_l],$$

where $j_s < k_s$, $1 \le s \le l$, and $k_1 < k_2 < \cdots < k_l$.

The formula (10.4) was stated at first in [FK], Proposition 4.2.

Problem 10.4 *It is an interesting task to find the Koszul dual of the algebra $\mathcal{A}_n^t$.*

Let us postpone a proof of Theorem 10.2 to the end of this Section, and start with a review of some results obtained by V.I. Arnold [A], I.M. Gelfand and A.N. Varchenko [GV].

It seems to be an interesting problem to understand the connections between the algebra $\mathcal{E}_n^0$ and the Orlik–Solomon algebra [OS] that corresponds to the Coxeter hyperplane arrangement (of type A_{n-1}). Let us remind that V.I. Arnold [A] described this algebra as the quotient of the exterior algebra of the vector space spanned by the $[ij]$ by the ideal generated by the relations (10.2). The "even analog" of the Orlik–Solomon algebra of a hyperplane arrangement was introduced and studied by I.M. Gelfand and A.N. Varchenko in [GV]. For the Coxeter hyperplane arrangement of type A_{n-1}, the results of Theorems 5 and 6 of [GV] can be formulated as follows.

Let $\mathcal{P}_n$ be the ring of integer–valued functions that are defined on the complement M_C of the union of a finite set C of hyperplaines in real, n–dimensional affine space, that have constant values on each connected component. The ring $\mathcal{P}_n$ has a

distinguish set of generators, namely, the Heaviside functions of hyperplaines: for each hyperplane $f(x) = 0$, the Heaviside function of the hyperplane f takes the value 1 from one side (where $f(x) > 0$), and 0 from the other (where $f(x) < 0$). Any element of the ring $\mathcal{P}_n$ is a polynomial in the Heaviside functions. Now let us assume that

$$\mathcal{M}_C = \mathcal{R}^n \setminus \bigcup_{1 \le i < j \le n} \{z_i - z_j = 0\}.$$

Let X_{ij} be the Heaviside function of the hyperplane $H_{ij} = \{z_i - z_j = 0\} \subset \mathcal{R}^n$, then ([GV], Theorems 5, 6, and 8)

• The Heaviside functions X_{ij}, $1 \le i < j \le n$, satisfy the following relations:

$$X_{ij}^2 - X_{ij} = 0, \tag{10.5}$$

$$X_{ij}(X_{ik} - 1)X_{jk} - (X_{jk} - 1)X_{ik}(X_{ij} - 1) = 0, \tag{10.6}$$

for any $1 \le i < j < k \le n$.

• Let $\mathcal{T}_n$ be an ideal in the ring of polynomials $\mathbb{Z}[X_{ij},\ 1 \le i < j \le n]$ generated by the left–hand sides of the relations (10.5) and (10.6). Then the natural homomorphism $\mathbb{Z}[X_{ij}]/\mathcal{T}_n \to \mathcal{P}_n$ is an isomorphism.

• Let us define the degree of a function $p \in \mathcal{P}_n$ as the minimum of the degrees of polynomials in $\{X_{ij}\}$ that represent p, and denote by $\mathcal{P}_n^k$ the subspace of functions degree not higher then k. Then there exists a noncanonical linear map

$$\pi_k \ : \ \mathcal{P}_n^k \to H^k(M_C, \mathbb{Z}),$$

which induces an isomorphism ($\mathcal{P}_n^{-1} = \phi$)

$$\mathcal{P}_n^k/\mathcal{P}_n^{k-1} \xrightarrow{\sim} H^k(M_C, \mathbb{Z}),\ 0 \le k \le n.$$

As a corollary, we have

$$H(\mathcal{P}_n; t) := \sum_{k=0}^{n} t^k \dim \mathcal{P}_n^k/\mathcal{P}_n^{k-1} = \prod_{j=1}^{n-1} (1 + jt). \tag{10.7}$$

Now we are ready to prove Theorem 10.2.

Proof of Theorem 10.2. Let us rewrite the relations (10.5) and (10.6) in slightly different form

$$X_{ij}^2 = \beta X_{ij}, \quad 1 \le i < j \le n; \tag{10.8}$$

$$X_{ij}X_{jk} = X_{jk}X_{ik} + X_{ik}X_{ij} - \beta X_{ik}, \quad 1 \le i < j < k \le n, \tag{10.9}$$

where β is an additional deformation parameter.

We define a deformation $\mathcal{P}_{n,\beta}$ of the algebra $\mathcal{P}_n$ as the quotient algebra

$$\mathbb{Z}[X_{ij}]/\mathcal{T}_{n,\beta} = \mathcal{P}_{n,\beta},$$

where $\mathcal{I}_{n,\beta}$ is an ideal in the commutative ring of polynomials $\mathbb{Z}[X_{ij}]$ generated by the relations (10.8) and (10.9).

It is clear that $\mathcal{P}_{n,\beta=0} \cong \mathcal{A}_n^0$, and $\mathcal{P}_{n,\beta}$ is a flat deformation of $\mathcal{A}_n^0$. Thus, we have

$$\prod_{j=1}^{n-1}(1+jt) = H(\mathcal{P}_n;t) = H(\mathcal{P}_{n,\beta};t) = H(\mathcal{P}_{n,\beta=0};t) = H(\mathcal{A}_n^0;t).$$

■

Let us introduce a noncommutative version of the algebra $\mathcal{P}_{n,\beta}$.

Definition 10.5 *Define the algebra $\mathcal{L}_{n,\beta}$ (of type A_{n-1}) as the quadratic algebra over $\mathbb{Z}[\beta]$ with generators $[ij]$, $1 \le i < j \le n$, which satisfy the following relations*

$$\begin{aligned} &[ij][jk] = [jk][ik] + [ik][ij] + \beta[ik]\,, \\ &[jk][ij] = [ik][jk] + [ij][ik] + \beta[ik], \quad \textit{for } i < j < k\,; \end{aligned} \tag{10.10}$$

$$\begin{aligned} &[ij][kl] = [kl][ij], \\ &\textit{whenever } \{i,j\} \cap \{k,l\} = \phi\,, i < j, \textit{ and } k < l\,. \end{aligned} \tag{10.11}$$

The algebra $\mathcal{L}_{n,\beta}$ is a smooth deformation of the algebra $\mathcal{G}_n$, and has the same Hilbert series:

$$H(\mathcal{L}_{n,\beta};t) := \sum_{k=0}^{\infty} t^k \dim \mathcal{L}_{n,\beta}^{(k)}/\mathcal{L}_{n,\beta}^{(k-1)} = H(\mathcal{G}_n;t).$$

We can define also the quotients $\mathcal{L}_{n,\beta}^t$, $\mathcal{L}_{n,\beta}^0$, and commutative subalgebras $\mathcal{H}_{n,\beta}$, $\mathcal{H}_{n,\beta}^t$ and $\mathcal{H}_{n,\beta}^0$ generated by the Dunkl elements θ_j (see (5.1)), $1 \le j \le n$.

It seems interesting to describe the relations in the commutative subalgebra $\mathcal{H}_{n,\beta}^t \subset \mathcal{L}_{n,\beta}^t$. The work is in progress and we hope to present our results in the nearest future.

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